Perfectly Proper Friendly Equilibria

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ABSTRACT

This paper introduces the perfectly proper friendly equilibrium, a strict refinement of the proper friendly equilibrium and of the perfectly proper equilibrium, and proves that every finite non-cooperative game with friends that satisfies certain conditions has at least one perfectly proper friendly Equilibrium.

Keywords: n-person game, non-cooperative game, friendly equilibrium, perfectly proper equilibrium.

Classification AMS: 90D06, 90D10, 90D99.

INTRODUCTION

For a strategy combination to be a plausible solution of a non-cooperative game it must be a Nash equilibrium (Nash 1951), but more refined concepts must be appealed to in order to determine a solution from among the multiplicity of Nash equilibria that generally exist. Marchi (1991) suggested a refinement in which the strategy played by each player *i* at an equilibrium mixed strategy profile $s = (s_i, s_{-i})$ is not only one that maximizes that player's expected payoff (conditional on the other players sticking to the strategies assigned them by this profile), but is also one that, among the set of such best responses to s_{-i}, maximizes the payoff of his or her best friend, or *first friendly successor*; and in case of there being more than one such strategy, it is one of those that also maximize the payoff of his or her second best friend, or *second friendly successor*; and so on. This solution concept, the friendly equilibrium, admits further refinement as the perfect friendly equilibrium (Marchi 1991), which also refines Selten's (1975) concept of perfect equilibrium; and as proper friendly equilibrium (Marchi 1991), which also refines Myerson's (1978) concept of proper equilibrium.

In this paper we present the perfectly proper friendly equilibrium (PPFE), which refines both the proper friendly equilibrium and the concept of perfectly proper equilibrium, according to which the players who lose most by unilateral deviation from an equilibrium should be the least likely to deviate (García - Jurado 1989).

In section 2 of this paper we establish notation and state relevant concepts and results due to Marchi (1991). In section 3 we introduce the new solution concept and prove some of its properties.

NOTATION AND BACKGROUND

Let Γ be a finite n-person non-cooperative game in normal form, $\Gamma = \{\Sigma_i, H_i, i \in N = \{1,...,n\}\}$ where Σ_i is the set of pure strategies of player i and $H_i : \Sigma = \prod_{i=1}^{n} \Sigma_i \to \Re$ is his or her *payoff function*; and let $\tilde{\Gamma} = \{S_i, H_i, i \in N\}$ be the mixed extension of Γ, where

$$S_i = \left\{ s_i \in \Re^{\Sigma_i} : \forall s_i \in \Sigma_i \quad \mathrm{s_i}(s_i) \geq 0, \quad \sum_{s_i \in \Sigma_i} \mathrm{s_i}(s_i) = 1 \right\}$$

and the expected payoff function $H_i : S = \prod_{i=1}^{n} S_i \to \Re$ is defined by $H_i(s) = \sum_{s \in \Sigma} H_i(s).s(s)$ where $s = (s_1,...,s_n) \in \Sigma$ and $s(s) = \prod_{i=1}^{n} s_i(s_i)$, with $s_i \in S_i$ for all i.

For each $i \in N$ we assume given a finite sequence $f_{(i)}^1 = i, f_{(i)}^2,..., f_{(i)}^{k_i}$ $(1 \leq k_i \leq n)$ of friendly successors of player i. For each $\bar{s} \in S$ we define the sets

$$\Psi_i^1(\bar{s}) = \left\{ s_i \in S_i : H_i(\bar{s}/s_i) \geq H_i(\bar{s}/s_i') \quad \mathrm{s}_i' \in S_i \right\}$$

$$\Psi_i^2(\bar{s}) = \left\{ s_i \in \Psi_i^1(\bar{s}) : H_{f_{(i)}^2}(\bar{s}/s_i) \geq H_{f_{(i)}^2}(\bar{s}/s_i') \quad \mathrm{s}_i' \in \Psi_i^1(\bar{s}) \right\} \qquad (1)$$

$\vdots$

$$\Psi_i^{k_i}(\bar{s}) = \left\{ s_i \in \Psi_i^{k_i-1}(\bar{s}) : H_{f_{(i)}^{k_i}}(\bar{s}/s_i) \geq H_{f_{(i)}^{k_i}}(\bar{s}/s_i') \quad \mathrm{s}_i' \in \Psi_i^{k_i-1}(\bar{s}) \right\}$$

Definition 1 *Given* Γ *and sequences* $f_{(i)}^1, f_{(i)}^2,..., f_{(i)}^{k_i}$ *as above, a mixed strategy profile* $s \in S$ *is a friendly equilibrium if* $s_i \in \Psi_i^{k_i}(s)$ *for all* $i \in N$.

Theorem 1 *(Marchi 1991). Given* Γ *and sequences* $f_{(i)}^1, f_{(i)}^2, \dots, f_{(i)}^{k_i}$ *such that, for all i, the point-to-set correspondence* $\Psi_i^{k_i}$ *defined by equations (1) is upper semicontinuous, then there exists a friendly equilibrium.*

The concepts of e-perfect, e-proper, perfect and proper equilibria are made friendly in ways analogous to that in which the Nash equilibrium is made friendly by Definition 1, as follows.

Definition 2. *Given* Γ *and sequences* $f_{(i)}^1, f_{(i)}^2, \dots, f_{(i)}^{k_i}$ *and* $e > 0$ *a mixed strategy profile* $s \in S$ *is an* e*-perfect friendly equilibrium if it is completely mixed (i.e. if* $\forall i \in N$ *and* $s_i \in \Sigma_i, s_i(s_i) > 0$*) and, for all* $i \in N$*, satisfies the conditions:*

$$\forall s_i, s'_i \in \Sigma_i \qquad H_i(s/s_i) < H_i(s/s'_i) \Rightarrow s_i(s_i) \le e,$$

$$\forall s_i, s'_i \in \Psi_i^1(s) \qquad H_{f_{(i)}^2}(s/s_i) < H_{f_{(i)}^2}(s/s'_i) \Rightarrow s_i(s_i) \le e,$$

$$\vdots$$

$$\forall s_i, s'_i \in \Psi_i^{k_i-1}(s) \qquad H_{f_{(i)}^{k_i}}(s/s_i) < H_{f_{(i)}^{k_i}}(s/s'_i) \Rightarrow s_i(s_i) \le e.$$

and $s \in S$ is an e-proper friendly equilibrium if it is completely mixed and, for all $i \in N$, satisfies the conditions:

$$\forall s_i, s'_i \in \Sigma_i \qquad H_i(s/s_i) < H_i(s/s'_i) \Rightarrow s_i(s_i) \le e.s_i(s'_i),$$

$$\forall s_i, s'_i \in \Psi_i^1(s) \qquad H_{f_{(i)}^2}(s/s_i) < H_{f_{(i)}^2}(s/s'_i) \Rightarrow s_i(s_i) \le e.s_i(s'_i),$$

$$\vdots$$

$$\forall s_i, s'_i \in \Psi_i^{k_i-1}(s) \qquad H_{f_{(i)}^{k_i}}(s/s_i) < H_{f_{(i)}^{k_i}}(s/s'_i) \Rightarrow s_i(s_i) \le e.s_i(s'_i),$$

Definition 3. *Given* Γ *and sequences* $f_{(i)}^1, f_{(i)}^2, \dots, f_{(i)}^{k_i}$*, a mixed strategy profile* $\bar{s} \in S$ *is a perfect (proper) friendly equilibrium if there exist sequences* $\{e_k\}_{k \in N}$ *and* $\{s^k\}_{k \in N}$ *such that:*

$$\forall k \in N \;\; e_k > 0, \quad \lim_{k \to \infty} e_k = 0$$

$\forall k \in N \quad s^k$ is an e_k- perfect (proper) friendly equilibrium.

$$\lim_{k \to \infty} s^k = \bar{s}$$

Remark 1 If for each i $k_i = 1$, the friendly concepts defined in Definitions 1, 2 and 3 reduce to the corresponding "friendless" concepts of Nash equilibrium, e-perfect equilibrium, e - proper equilibrium, perfect equilibrium and proper equilibrium.

Remark 2. It is clear from Definition 3 that all proper friendly equilibria are also perfect friendly equilibria, but the converse does not necessarily hold.

Perfectly proper friendly equilibria

Definition 4. *Given a finite n-person non-cooperative game in normal form* Γ, *sequences* $f_{(i)}^1, f_{(i)}^2,..., f_{(i)}^{k_i}$ *and* $e > 0$, *then* $s \in S$ *s an* e *- perfectly proper friendly equilibrium* e *- PPFE) if it is completely mixed and, for all* $i, j \in N$ *satisfies the following conditions:*

- $\forall s_i \in \Psi_i^1(s), \forall s_j \in \Psi_j^1(s), \forall s_i' \in \Sigma_i, \forall \bar{s}_j \in \Sigma_j$

if $H_i(s/s_i) - H_i(s/s'_i) < H_j(s/s_j) - H_j\left(s/\bar{s}_j\right) \Rightarrow s_j\left(\bar{s}_j\right) \le e.s_i(s'_i);$

- $\forall s_i \in \Psi_i^2(s), \forall s_j \in \Psi_j^2(s), \forall s_i' \in \Psi_i^1(s), \forall \bar{s}_j \in \Psi_j^1(s)$

if $H_{f_{(i)}^2}(s/s_i) - H_{f_{(i)}^2}(s/s'_i) < H_{f_{(j)}^2}(s/s_j) - H_{f_{(j)}^2}\left(s/\bar{s}_j\right) \Rightarrow s_j\left(\bar{s}_j\right) \le e.s_i(s'_i);$

⋮

- $\forall s_i \in \Psi_i^{k_i}(s), \forall s_j \in \Psi_j^{k_j}(s), \forall s_i' \in \Psi_i^{k_i-1}(s), \forall \bar{s}_j \in \Psi_j^{k_j-1}(s)$

if $H_{f_{(i)}^{k_i}}(s/s_i) - H_{f_{(i)}^{k_i}}(s/s'_i) < H_{f_{(j)}^{k_j}}(s/s_j) - H_{f_{(j)}^{k_j}}\left(s/\bar{s}_j\right) \Rightarrow s_j\left(\bar{s}_j\right) \le e.s_i(s'_i);$

If the friendly successors of *i* are $f_{(i)}^1, f_{(i)}^2,..., f_{(i)}^m$, and those of *j* $f_{(j)}^1, f_{(j)}^2,..., f_{(j)}^n$ with $m < n$, then we define $f_{(i)}^r = f_{(i)}^m$ for all *r*, $r \in \{m+1,...,n\}$.

Remark 3. *For i = j, Definition 4 is the definition of an* e *- proper friendly equilibrium.*

Remark 4. *If* $k_i = 1$ *for all i, Definition 4 is the definition of an* e *- perfectly proper equilibrium.*

Definition 5. *Given a finite n-person non-cooperative game in normal form* Γ *and sequences* $f_{(i)}^1, f_{(i)}^2,..., f_{(i)}^{k_i}$, *a mixed strategy profile* $\bar{s} \in S$ *is a perfectly proper friendly equilibrium (PPFE) if there exist sequences* $\{e_k\}_{k\in N}$ *and* $\{s^k\}_{k\in N}$ *such that:*

$\forall k \in N \;\; e_k > 0, \quad \lim_{k\to\infty} e_k = 0$

$\forall k \in N \quad s^k$ *is an* e_k*- perfect (proper) friendly equilibrium.*

$\lim_{k\to\infty} s^k = \bar{s}$

Theorem 2.

a) All perfectly proper friendly equilibria are perfectly proper equilibria.

b) All perfectly proper friendly equilibria are proper friendly equilibria.

Proof By the definitions and Remarks 3 and 4.

The following examples show that the converses of the statements of Theorem 2 do not necessarily hold.

Example Let player 2 be the friendly successor of player 1 in the following game:

	b_1	b_2	b_3
a_1	2 2	1 0	1 0
a_2	2 2	0 1	2 1
a_3	1 2	1 301	1 2
a_4	1 2	1 1	1 3
a_5	2 2	1 0	1 0

In this game (a_2, b_1) and, for all $m \in [0,1]$, $(m a_1 + (1-m)a_5, b_1)$ are all perfectly proper equilibria, but only (a_2, b_1) is a PPFE, as may be seen by considering, for (a_2, b_1), the sequences $\{e_k\}_{k\in N}$ and $\{s^k\}_{k\in N}$ defined by

$$\forall k \quad e_k = \frac{1}{k+2}.$$

$$s_1^k(a_1) = \frac{1}{2(k+2)}, \quad s_1^k(a_3) = \frac{1}{300(k+2)^3}, \quad s_1^k(a_4) = \frac{1}{2(k+2)^3}, \quad s_1^k(a_3) = \frac{1}{2(k+2)^2}$$

$$s_1^k(a_2) = 1 - \frac{150(k+2)^2 + 150(k+2) + 151}{300(k+2)^3}.$$

$$s_2^k(b_1) = 1 - \frac{1+(k+2)^2}{150(k+2)^7}, \quad s_2^k(b_2) = \frac{1}{150(k+2)^7}, \quad s_2^k(b_3) = \frac{1}{150(k+2)^5}.$$

And for $(m a_1 + (1-m)a_5, b_1)$ the sequences defined by

$$\forall k \quad e_k = \frac{1}{k+2}.$$

$$s_1^k(a\ ) = \frac{1}{(k+2)}, \quad s_1^k(a_3) = s_1^k(a_4) = \frac{1}{(k+2)[300(k+2)-1]},$$

$$s_1^k(a_1) = s_1^k(a_5) = \frac{1}{2}\left[1 - \frac{1}{k+2} - \frac{2}{(k+2)[300(k+2)-1]}\right].$$

$$s_2^k(b_1) = 1 - \frac{k+3}{(k+2)^3[300(k+2)-1]}, \quad s_2^k(b_2) = \frac{1}{(k+2)^2[300(k+2)-1]},$$

$$s_2^k(b_3) = \frac{1}{(k+2)^3[300(k+2)-1]}$$

Example. Let player 2 be the friendly successor of player 1 in the following 3-player game:

	a_1^2	a_1^2
a_1^1	1, 1, 1	0, 0, 1
a_2^1	0, 0, 1	0, 0, 1

a_1^3

	a_1^2	a_1^2
a_1^1	0, 0, 0	0, 0, 0
a_2^1	0, 0, 0	1, 1, 0

In this game $\left(a_2^1, a_1^3\right)$ and $\left(a_1^1, a_1^2, a_1^3\right)$ are both proper friendly equilibria, but $\left(a_2^1, a_2^2, a_1^3\right)$ is not a PPFE because it is not perfectly proper (see García-Jurado (1989)).

Theorem 3

Every finite non-cooperative game Γ *with friends for which the* $\Psi_i^r(s)$ *are upper semicontinuous has at least one PPFE.*

Proof.

Let $g = \dfrac{e^{(a_1 + a_2 + \cdots + a_r)\sum_{i=1}^{n} m_i}}{\sum_{i=1}^{n} m_i}$ where $e \in (0,1)$, $m_i = |\Sigma_i|$ and $a_k = \left(1 + \sum_{i=1}^{n} m_i\right)$ $(1 \le k \le r)$.

$\forall i$, let $S_i(g) = \{s_i \in S_i : \forall s_{-i} \in \Sigma_{-i} \quad s_i(s_{-i}) \geq g\}$; and let $S(g) = S_1(g) \times \cdots \times S_n(g)$. $S(g)$ is compact, convex and non-empty

Consider the correspondence $F : S(g) \to P(S(g))$ such that, $\forall s' \in S(g)$

$F(s') = \{s \in S(g) \mid \forall i, j \in N$

$a)\ \forall s_{-i} \in \Psi_i^1(s'), \forall s_{-j} \in \Psi_j^1(s'), \forall s_{-i}^{'} \in \Sigma_{-i}, \forall \bar{s}_{-j} \in \Sigma_{-j}$

if $H_i(s'/s_{-i}) - H_i(s'/s'_{-i}) < H_j(s'/s_{-j}) - H_j\left(s'/\bar{s}_{-j}\right) \Rightarrow s_j\left(\bar{s}_{-j}\right) \leq e.s_i(s'_{-i})$;

$b)\ \forall s_{-i} \in \Psi_i^2(s), \forall s_{-j} \in \Psi_j^2(s), \forall s_{-i}^{'} \in \Psi_i^1(s), \forall \bar{s}_{-j} \in \Psi_j^1(s)$

if $H_{f_{(i)}^2}(s/s_{-i}) - H_{f_{(i)}^2}(s/s'_{-i}) < H_{f_{(j)}^2}(s/s_{-j}) - H_{f_{(j)}^2}\left(s/\bar{s}_{-j}\right) \Rightarrow s_j\left(\bar{s}_{-j}\right) \leq e.s_i(s'_{-i})$;

$\vdots$

$r)\ \forall s_{-i} \in \Psi_i^r(s), \forall s_{-j} \in \Psi_j^r(s), \forall s_{-i}^{'} \in \Psi_i^{r-1}(s), \forall \bar{s}_{-j} \in \Psi_j^{r-1}(s)$

if $H_{f_{(i)}^r}(s/s_{-i}) - H_{f_{(i)}^r}(s/s'_{-i}) < H_{f_{(j)}^r}(s/s_{-j}) - H_{f_{(j)}^r}\left(s/\bar{s}_{-j}\right) \Rightarrow s_j\left(\bar{s}_{-j}\right) \leq e.s_i(s'_{-i})\Big\}$;

We shall show that F complies with the requirements of the Kakutani fixed point theorem. We first note that $\forall s \in S(g)$, $F(s)$ is convex and compact. We now show that $\forall s \in S(g)$, $F(s) \neq \emptyset$.

Let $s \in S(g)$. $\forall i \in N$, $s'_{-i} \in \Sigma_{-i}$ we define

$$A_i(s/s'_{-i}) = \sum_{j=1}^{n} \left| \{\bar{s}_{-j} \in \Sigma_{-j} / \forall s_{-i} \in \Psi_i^1(s), \forall s_{-j} \in \Psi_j^1(s)\ H_j(s/s_{-j}) - H_j(s/\bar{s}_{-j}) < H_i(s/s_{-i}) - H_i(s/s'_{-i})\} \right|$$

and $\forall k, k = \{2,...,r\}$, $\forall s'_{-i} \in \Psi_i^{k-1}(s)$ we define

$$A_{f_{(i)}^k}(s/s'_{-i}) = \sum_{j=1}^{n} \Big| \{\bar{s}_{-j} \in \Psi_j^{k-1}(s) / \forall s_{-i} \in \Psi_i^k(s), \forall s_{-j} \in \Psi_j^k(s)$$

$$H_{f_{(j)}^k}(s/s_{-j}) - H_{f_{(j)}^k}(s/\bar{s}_{-j}) < H_{f_{(i)}^k}(s/s_{-i}) - H_{f_{(i)}^k}(s/s'_{-i})\} \Big|$$

Consider $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$, defined by

$$\bar{s}_i(s_{-i}) = \frac{e^{\sum_{k=1}^{r} a_k A_{f_{(i)}^k}(s/s_{-i})}}{\sum_{u=1}^{n} \sum_{s'_{-u} \in \Sigma_{-u}} e^{\sum_{k=1}^{r} a_k A_{f_{(u)}^k}(s/s'_{-u})}} \quad \forall s_{-i} \notin \Psi_i^r(s)$$

$$\bar{s}_i(s_{-i}) = \frac{1 - \sum_{s'_{-i} \notin \Psi_i^r(s)} \bar{s}_i(s'_{-i})}{\left|\Psi_i^r(s)\right|} \quad \forall s_{-i} \in \Psi_i^r(s).$$

$\bar{s} \in S$, because $\forall i \in N, s_i \in \Sigma_i$

$\bar{s}_i(s_i) \geq 0$, and $\sum_{s_i \in \Sigma_i} \bar{s}_i(s_i) = 1$

Moreover, $\bar{s} \in S(g)$

i.e. $\forall\ s_i \in \Sigma_i\ \ \bar{s}_i(s_i) \geq g$: for

a) if $\tilde{s}_i \notin \Psi_i^r(s)$, then $\bar{s}_i(s_i) \geq g$ follows from

$$\sum_{k=1}^{r} a_k A_{f_{(i)}^k}(s/s_i) \leq \sum_{k=1}^{r} a_k \sum_{i=1}^{n} m_i$$

and

$$\sum_{u=1}^{n} \sum_{s'_u} e^{\sum_{k=1}^{r} a_k A_{f_{(i)}^k}(s/s_i)} \leq \sum_{u=1}^{n} m_u$$

and b) if $\tilde{s}_i \in \Psi_i^r(s)$, then $\bar{s}_i(s_i) \geq g$ follows from

$$1 - \sum_{s \notin \Psi_{(i)}^r(s)} \bar{s}_i(s_i) \geq \left|\Psi_i^r(s)\right|.g\,.$$

We now show that $\bar{s} \in F(s)$.

$\forall t,\quad t \in \{1,...,r\}$, consider any $\tilde{s}_i \in \Psi_i^{t-1}(s), \bar{s}_j \in \Psi_j^{t-1}(s)$ such that $\forall\ s_i \in \Psi_i^t(s), s_j \in \Psi_j^t(s)$

$$H_{f_{(i)}^k}(s/s_i) - H_{f_{(i)}^k}(s/\tilde{s}_i) < H_{f_{(j)}^k}(s/s_j) - H_{f_{(j)}^k}(s/\bar{s}_j)$$

Clearly, $\bar{s}_j \notin \Psi_j^t(s)$ and $A_{f_{(j)}^t}(s/\bar{s}_j) \geq 1 + A_{f_{(i)}^t}(s/\tilde{s}_i)$.

Now either a) $\tilde{s}_i \notin \Psi_i^t(s)$ or b) $\tilde{s}_i \in \Psi_i^t(s)$.

a) If $\tilde{s}_i \notin \Psi_i^t(s)$ then $\tilde{s}_i \notin \Psi_i^r(s)$, whence

$$\bar{s}_j(\bar{s}_j) \leq e\,\frac{e^{(a_t - 1) + a_t A_{f_{(i)}^t}(s/\tilde{s}_i)}}{\sum_{u=1}^{n} \sum_{s'_u \in \Sigma_u} e^{\sum_{k=1}^{r} a_k A_{f_{(u)}^k}(s/s'_u)}} \leq e\,\frac{e^{a_t A_{f_{(i)}^t}(s/\tilde{s}_i) + a_{t+1} A_{f_{(i)}^{t+1}}(s/\tilde{s}_i) + \cdots + a_r A_{f_{(i)}^r}(s/\tilde{s}_i)}}{\sum_{u=1}^{n} \sum_{s'_u \in \Sigma_u} e^{\sum_{k=1}^{r} a_k A_{f_{(u)}^k}(s/s'_u)}}$$

and since $A_{f_{(i)}^t}(s/\tilde{s}_i) = 0$ when $\tilde{s}_i \in \Psi_i^{t-1}(s)$

$$\bar{s}_j(\bar{s}_j) \leq e.\bar{s}_i(\tilde{s}_i).$$

b) If $\tilde{s}_i \in \Psi_i^t(s)$, then either

i) $\exists\, p \in \{t+1,...,r\}$ such that $\tilde{s}_i \notin \Psi_i^p(s)$, in which case an argument analogous that developed in a) leads to the conclusion $\bar{s}_j(\bar{s}_j) \leq e.\bar{s}_i(\tilde{s}_i)$;

or ii) $\tilde{s}_i \in \Psi_i^r(s)$, in which case $A_{f_{(i)}^r}(s/\tilde{s}_i) = \cdots = A_i(s/\tilde{s}_i) = 0$, whence $\bar{s}_j(\bar{s}_j) \leq e.\bar{s}_i(\tilde{s}_i)$.

Hence $\bar{\bar{s}}_j\left(\bar{s}_{-j}\right) \leq e.\bar{s}_i\left(\tilde{\hat{s}}_{-i}\right)$, whence $\bar{s} \in F(s)$ and $F(s)$ is non-empty.

Since F is also upper semicontinuous (because, by hypothesis, the $\Psi_i^k(s)$ are), the conditions of the Kakutani fixed point theorem are satisfied and F has a fixed point, which is therefore an e -PPFE. Hence, given $\{e_k\}_{k \in N}$ such $\lim_{k \to \infty} e_k = 0$, there is a sequence of e-PPFE $\{s^k\}_{k \in N}$ and since S is compact, this sequence has a limit point, which is therefore a PPFE.

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